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VIBRATIONS OF HIGHLY PRESTRESSED ANISOTROPIC PLATES VIA A NUMERICAL-PERTURBATION TECHNIQUE

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Abstract—The method of matched asymptotic expansions is used to reduce the problem of the transverse vibrations of a highly prestressed anisotropic plate into the simpler problem of the vibration of an anisotropic membrane with modified boundary conditions that account for the bending effects. In the absence of an exact solution the membrane problem can be solved by any well-known numerical technique. The numerical-perturbation results for a clamped circular plate with rectangular orthotropy and a uniform tensile stress applied on its boundary show an excellent correlation with finite-element solutions for the original problem. Furthermore, the solutions obtained for annular plates form the basis for solutions to problems involving near-annular plates.

1. INTRODUCTION

For a generally anisotropic circular plate, the material rigidities vary with the polar angle as well as the radial direction. Closed form solutions for the transverse vibrations of such plates under the action of general in-plane forces do not exist, and recourse has to be made to approximate techniques such as numerical methods, perturbation methods, or combinations of both. Exact solutions to the problem of the transverse vibrations of a uniformly prestressed isotropic plate of circular or annular geometry are well-known[1]. The solutions of the problem under consideration could be reduced to these known cases for comparison purposes. Approximate solutions for anisotropic plates of regular geometries have been obtained by using Rayleigh–Ritz and Galerkin techniques[2].

When the applied loads are very large compared with D_{11}^*/a^{*2}, where D_{11}^* is the characteristic rigidity and a^* is a characteristic dimension of the plate, a state of membrane deformation prevails over the plate except in a thin layer next to the edge of the plate where a state of bending deformation exists. If $\epsilon = (D_{11}^*/N^*a^{*2})^{1/2} \ll 1$, where N^* is a measure of the applied inplane loads, the bending layer becomes thinner as ϵ decreases.

When a purely numerical technique, such as the finite-element method[3], is attempted, the entire plate is modelled by using a large number of bending elements to accurately simulate the effects of the boundary conditions on the displacement and stress distributions, thereby not exploiting the fact that a large portion of the plate behaves like a prestressed membrane. As $\epsilon \to 0$, more and more elements will be needed, resulting in a larger computational time. A numerical-perturbation technique is proposed as an economical alternative for the treatment of the vibrations of highly prestressed anisotropic plates of complex geometries. In the present paper we develop the solutions for circular plates, which would form the basis for plates of more complex geometries.

The perturbation method of matched asymptotic expansions[4] is used to reduce the solution of the problem of the transverse vibrations of highly prestressed anisotropic plates to the solution of the simpler problem, the transverse vibrations of membranes, but with modified boundary conditions that account for the effects of bending. This simpler problem could then be solved by using a numerical technique like the finite-element method. Since the membrane has no bending rigidity only the geometric matrix is needed in a membrane element, while the stiffness as well as the geometric matrices are needed in a bending element to model the effects of in-plane forces on out-of-plane deformations. Moreover, since the governing equation in the case of a vibrating membrane is a second-order partial differential equation, the geometric matrix for this analysis can be based upon a lower-order polynomial for the out-of-plane displacement function than that which would be used for vibrating plates. This then leads to a reduction in the size of the geometric matrix for each element, which in turn implies a reduction of the total number of degrees of freedom of the assembled model. This technique has been proposed for the problem of the transverse vibrations of highly prestressed variable thickness plates [5].

2. PROBLEM FORMULATION

We consider linear transverse vibrations of a highly prestressed, midplane-symmetric, circular plate with rectangular orthotropy. Since six independent elastic constants are required for its constitutive description, such a plate is anisotropic. Midplane symmetry eliminates bending-extensional coupling in the plate. The transverse vibrations are assumed to be small enough for the effects of midplane stretching on the inplane loads to be neglected.

We introduce the radius of the plate a^*, thickness of the plate h^*, a characteristic period $T^* = (\rho^* h^* a^{*4}/D_{ij}^*)^{1/2}$ and a characteristic load N^* as reference quantities for writing down the governing equation in dimensionless form. Thus, if starred and unstarred quantities denote dimensional and dimensionless quantities respectively, we have:

$$w = w^*/a^*, \quad t = t^*/T^*, \quad r = r^*/a^*, \quad h_k = h_k^*/h^*, \quad N_{rr} = N_{rr}^*/N^* \tag{1}$$

$$N_{\theta\theta} = N_{\theta\theta}^*/N^*, \quad N_{r\theta} = N_{r\theta}^*/N^*, \quad D_{ij} = D_{ij}^*/(N^* h^{*2}), \quad q = q^* a^*/N^* \tag{2}$$

where w is the transverse displacement, t is the time, r is the radial distance, θ is the circumferential coordinate, $(h_k - h_{k-1})$ is the thickness of the kth layer in the plate, the N_{ij} are the inplace load distributions, the D_{ij} are the bending rigidities in rectangular form [2], and q is the transverse loading on the plate. Using the bending moment-curvature relations [2] in the equation describing the transverse vibrations of an anisotropic plate [6], and introducing the dimensionless quantities, we obtain:

$$\epsilon^2 \mathscr{L}_1(w) = \mathscr{L}_2(w) - \alpha^2 w_{,tt} + q \tag{3}$$

where

$$\begin{aligned}\mathscr{L}_1 = \Bigg\{ & \frac{\bar{D}_{11}}{D_{11}}\frac{\partial^4}{\partial r^4} + \frac{4}{r}\frac{\bar{D}_{16}}{D_{11}}\frac{\partial^4}{\partial r^3 \partial\theta} + \frac{2}{r^2}\frac{(\bar{D}_{12}+2\bar{D}_{66})}{D_{11}}\frac{\partial^4}{\partial r^2\partial\theta^2} + \frac{4}{r^3}\frac{\bar{D}_{26}}{D_{11}}\frac{\partial^4}{\partial r\partial\theta^3} + \frac{1}{r^4}\frac{\bar{D}_{22}}{D_{11}}\frac{\partial^4}{\partial\theta^4} + \frac{2}{r}\frac{\bar{D}_{11}}{D_{11}}\frac{\partial^3}{\partial r^3} \\ & - \frac{2}{r^3}\frac{(\bar{D}_{12}+2\bar{D}_{66})}{D_{11}}\frac{\partial^3}{\partial r\partial\theta^2} - \frac{4}{r^4}\frac{\bar{D}_{26}}{D_{11}}\frac{\partial^3}{\partial\theta^3} - \frac{1}{r^2}\frac{\bar{D}_{22}}{D_{11}}\frac{\partial^2}{\partial r^2} + \frac{4}{r^3}\frac{(\bar{D}_{16}+\bar{D}_{26})}{D_{11}}\frac{\partial^2}{\partial r\partial\theta} \\ & + \frac{2}{r^4}\frac{(\bar{D}_{11}+\bar{D}_{22}+2\bar{D}_{66})}{D_{11}}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^3}\frac{\bar{D}_{22}}{D_{11}}\frac{\partial}{\partial r} - \frac{4}{r^4}\frac{(\bar{D}_{16}+\bar{D}_{26})}{D_{11}}\frac{\partial}{\partial\theta}\Bigg\}\end{aligned} \tag{4}$$

$$\mathscr{L}_2 = \left\{ N_{rr}\left(\frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2}\right) + N_{rr,r}\frac{\partial}{\partial r} + \frac{N_{\theta\theta}}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{1}{r^2}N_{\theta\theta,\theta}\frac{\partial}{\partial\theta} + \frac{2}{r}N_{r\theta}\frac{\partial^2}{\partial r\partial\theta} + \frac{1}{r}N_{r\theta,r}\frac{\partial}{\partial\theta} + \frac{1}{r}N_{r\theta,\theta}\frac{\partial}{\partial r}\right\} \tag{5}$$

$$\epsilon^2 = D_{11}^*/(a^{*2}N^*); \qquad \alpha^2 = \rho^* a^{*2} h^*/N^* T^{*2} \tag{6}$$

where $\rho^* h^*$ is the mass per unit area of the plate and $\bar{D}_{ij}$ are transformed bending rigidities in polar form. The inplane loads N_{rr}, $N_{\theta\theta}$ and $N_{r\theta}$ are assumed to be known functions of r and θ. These are the solutions of the plane stress problem which may be obtained numerically for any given loading conditions.

To complete the problem formulation we specify the boundary conditions. For definiteness we consider the case of a clamped circular plate for which

$$w = 0 \quad \text{and} \quad \frac{\partial w}{\partial r} = 0 \quad \text{at} \quad r = 1 \tag{7}$$

$$w < \infty. \tag{8}$$

3. REDUCTION TO A MEMBRANE PROBLEM

Outer expansion

We seek an approximate solution, a so-called outer expansion or a membrane expansion as described below, in the form:

$$w^{(m)} = w_0^{(m)}(r, \theta, t) + \epsilon w_1^{(m)}(r, \theta, t) + \cdots. \tag{9}$$

Substituting eqn (9) into eqn (3) and equating coefficients of like powers of ϵ on both sides, we obtain

$$\mathscr{L}_2(w_0^{(m)}) = \alpha^2 w_{0,tt}^{(m)} - q \tag{10}$$

$$\mathscr{L}_2(w_1^{(m)}) = \alpha^2 w_{1,tt}^{(m)}. \tag{11}$$

We note that the equations governing the $w_n^{(m)}$ are of second order as opposed to the original eqn (3) which is of fourth order. Hence the expansion (9) cannot be expected to satisfy, in general, all the boundary conditions. Consequently, it is not valid near the plate edge. Physically, there is a thin layer near the edge where bending deformations exist, and a state of membrane deformation exists everywhere else for small ϵ values. Across these edge layers the displacement changes very rapidly from a membrane type to a bending type in order to satisfy the boundary conditions [7–9].

Inner expansion near r = 1

A bending expansion that is valid in the edge layer is obtained through the method of matched asymptotic expansions by using a stretching transformation:

$$\zeta = (1 - r)/\epsilon. \tag{12}$$

In this case we seek an expansion of the form:

$$w^{(i)} = w_0^{(i)}(\zeta, \theta, t) + \epsilon w_1^{(i)}(\zeta, \theta, t) + \cdots. \tag{13}$$

Writing the original eqn (3) in terms of the stretched variable ζ, substituting (13) for $w^{(i)}$, and collecting coefficients of powers of ϵ, we have:

$$\frac{\partial^4 w_0^{(i)}}{\partial \zeta^4} - \frac{D_{11}}{\bar{D}_{11}} N_{rr} \frac{\partial^2 w_0^{(i)}}{\partial \zeta^2} = 0 \tag{14}$$

$$\frac{\partial^4 w_1^{(i)}}{\partial \zeta^4} - \frac{D_{11}}{\bar{D}_{11}} N_{rr} \frac{\partial^2 w_1^{(i)}}{\partial \zeta^2} = 4\frac{\bar{D}_{16}}{\bar{D}_{11}} \frac{\partial^4 w_0^{(i)}}{\partial \zeta^2 \partial \theta^2} + 2\frac{\partial^3 w_0^{(i)}}{\partial \zeta^3} - \frac{D_{11}}{\bar{D}_{11}} \left\{ N_{rr} \frac{\partial w_0^{(i)}}{\partial \zeta} + N_{rr,r} \frac{\partial w_0^{(i)}}{\partial \zeta} + 2N_{r\theta} \frac{\partial^2 w_0^{(i)}}{\partial \zeta \partial \theta} + N_{r\theta,\theta} \frac{\partial w_0^{(i)}}{\partial \zeta} \right\}. \tag{15}$$

If we let $(D_{11}N_{rr})/(\bar{D}_{11}) = \mu^2(\theta)$, the general solution of (14) can be expressed as

$$w_0^{(i)}(\zeta, \theta, t) = \alpha_1(\theta, t) + \alpha_2(\theta, t)\zeta + \alpha_3(\theta, t) \exp(-\mu\zeta) + \alpha_4(\theta, t) \exp(\mu\zeta) \tag{16}$$

subject to the boundary conditions:

$$w_0^{(i)} = \partial w_0^{(i)}/\partial \zeta = 0 \quad \text{at} \quad \zeta = 0 \tag{17}$$

As $w_0^{(i)}$ is finite and cannot grow exponentially, $\alpha_4(\theta, t)$ must be zero. Using the boundary conditions (17) in eqn (16), we obtain:

$$w_0^{(i)}(\zeta, \theta, t) = \alpha_1(\theta, t)[1 - \mu(\theta)\zeta - \exp(-\mu(\theta)\zeta)]. \tag{18}$$

This can be substituted into eqn (15) to get the general solution for $w_1^{(i)}(\zeta, \theta, t)$. But we choose to apply the matching condition at this stage to evaluate $\alpha_1(\theta, t)$ which will be shown to be zero. Hence the solution for $w_1^{(i)}(\zeta, \theta, t)$ is obtained in a simpler fashion.

Matching

We match one-term inner expansion with two-term outer expansion by using the matching principle

1-term inner (2-term outer) = 2-term outer (1-term inner)

2-term outer = $w_0^{(m)}(r, \theta, t) + \epsilon w_1^{(m)}(r, \theta, t)$.

Rewritten in inner variable = $w_0^{(m)}(1-\epsilon\zeta, \theta, t) + \epsilon w_1^{(m)}(1-\epsilon\zeta, \theta, t)$.

Expanded for small ϵ = $w_0^{(m)}(1, \theta, t) + \epsilon[w_1^{(m)}(1, \theta, t) - \zeta w_0'^{(m)}(1, \theta, t)] + O(\epsilon^2)$.

1-term inner of this = $w_0^{(m)}(1, \theta, t)$ (19)

1-term inner = $\alpha_1[1 - \mu\zeta - \exp(-\mu\zeta)]$

Rewritten in outer variable = $\alpha_1\left[1 - \mu\dfrac{(1-r)}{\epsilon} - \exp[-\mu(1-r)/\epsilon]\right]$

Expanded for small ϵ = $\alpha_1[1 - \mu(1-r)\epsilon^{-1}]$

2-term outer of this = $-\alpha_1\mu(1-r)\epsilon^{-1} + \alpha_1$. (20)

Using the matching condition and equating coefficients of like powers of ϵ, we have:

$$-\alpha_1\mu(1-r) = 0; \quad \text{i.e.} \quad \alpha_1 = 0 \quad \text{from} \quad \epsilon^{-1} \quad \text{terms}$$

$$\alpha_1 = w_0^{(m)}(1, \theta, t) \quad \text{from } \epsilon^0 \text{ terms.}$$

Hence,

$$w_0^{(m)}(1, \theta, t) = \alpha_1 = 0. \tag{21}$$

Using this in eqn (18) we see that

$$w_0^{(i)}(\zeta, \theta, t) \equiv 0. \tag{22}$$

Substituting this result into equation (15), we get

$$\frac{\partial^4 w_1^{(i)}}{\partial\zeta^4} - \mu^2(\theta)\frac{\partial^2 w_1^{(i)}}{\partial\zeta^2} = 0. \tag{23}$$

The boundary conditions are

$$w_1^{(i)} = \partial w_1^{(i)}/\partial\zeta = 0 \quad \text{at} \quad \zeta = 0. \tag{24}$$

Equations (23) and (24) have the same type of solution as for $w_0^{(i)}$, namely,

$$w_1^{(i)}(\zeta, \theta, t) = \alpha_5(\theta, t)[1 - \mu(\theta)\zeta - \exp(-\mu(\theta)\zeta)] \tag{25}$$

So,

$$w^{(i)} = \epsilon\alpha_5(1 - \mu\zeta - \exp(-\mu\zeta)) + O(\epsilon^2). \tag{26}$$

Matching

Equation (21) gives the first modified boundary condition for the membrane problem. To get the second one, we use the following matching condition that includes $w_1^{(i)}$:

2-term inner (2-term outer) = 2-term outer (2-term inner)

2-term outer $= w_0^{(m)}(r, \theta, t) + \epsilon w_1^{(m)}(r, \theta, t)$.

Rewritten in inner variable $= w_0^{(m)}(1 - \epsilon\zeta, \theta, t) + \epsilon w_1^{(m)}(1 - \epsilon\zeta, \theta, t)$.

Expanded for small $\epsilon = w_0^{(m)}(1, \theta, t) + \epsilon[w_1^{(m)}(1, \theta, t) - \zeta w_0'^{(m)}(1, \theta, t)] + O(\epsilon^2)$.

$$\text{2-term inner (2-term outer)} = w_0^{(m)}(1, \theta, t) + \epsilon[w_1^{(m)}(1, \theta, t) - \zeta w_0'^{(m)}(1, \theta, t)]$$
$$= \epsilon[w_1^{(m)}(1, \theta, t) - \zeta w_0'^{(m)}(1, \theta, t)] \tag{27}$$

as $w_0^{(m)}(1, \theta, t) = 0$.

$$\text{2-term inner} = w_0^{(i)}(\zeta, \theta, t) + \epsilon w_1^{(i)}(\zeta, \theta, t)$$
$$= \epsilon w_1^{(i)}(\zeta, \theta, t) = \epsilon\alpha_5[1 - \mu\zeta - \exp(-\mu\zeta)].$$

Rewritten in outer variable $= \epsilon[1 - \mu(1 - r)/\epsilon - \exp(-\mu(1 - r)/\epsilon)]\alpha_5$.

Expanded for small $\epsilon = -\alpha_5\mu(1 - r) + \epsilon\alpha_5$

$$\text{2-term outer (2-term inner)} = -\alpha_5\mu(1 - r) + \epsilon\alpha_5. \tag{28}$$

Using the matching condition, expressing $\epsilon\zeta$ as $(1 - r)$ in eqn (27), and equating coefficients of like powers of ϵ, we obtain

$$-(1 - r)w_0'^{(m)}(1, \theta, t) = -\alpha_5\mu(1 - r)$$

or

$$w_0'^{(m)}(1, \theta, t) = \mu(\theta)\alpha_5(\theta, t) \quad \text{from } \epsilon^0 \text{ terms} \tag{29}$$

$$w_1^{(m)}(1, \theta, t) = \alpha_5(\theta, t) \quad \text{from } \epsilon \text{ terms.} \tag{30}$$

It follows from equations (29) and (30) that

$$w_1^{(m)}(1, \theta, t) = \mu^{-1}(\theta)w_0'^{(m)}(1, \theta, t) \tag{31}$$

Equations (21) and (31) are the modified boundary conditions subject to which the membrane solution $w^{(m)}$ is obtained from eqns (9)–(11).

Free vibrations

If free vibrations ($q = 0$) are considered and the solution is assumed to be harmonic with respect to time,

$$w^{(m)}(r, \theta, t; \epsilon) = \phi(r, \theta; \epsilon)\exp(i\omega t). \tag{32}$$

If we let

$$\phi(r, \theta) = \phi_0(r, \theta) + \epsilon\phi_1(r, \theta) + \cdots$$

eqns (9)–(11) yield

$$\mathcal{L}_2\phi_0 + \lambda^2\phi_0 = 0 \tag{33}$$

$$\mathcal{L}_2\phi_1 + \lambda^2\phi_1 = 0 \tag{34}$$

where

$$\lambda^2 = \alpha^2\omega^2. \tag{35}$$

Similarly the boundary conditions (21) and (31) become

$$\phi_0(1, \theta) = 0 \tag{36}$$

$$\phi_1(1, \theta) = \mu^{-1}(\theta)\phi_0'(1, \theta). \tag{37}$$

The homogeneous eqns (33) and (34) subject to the boundary conditions (36) and (37) can have only the trivial solution unless we expand the eigenvalue λ as a function of ϵ. That is,

$$\lambda = \lambda_0 + \epsilon\lambda_1 + \cdots. \tag{38}$$

This leads to

$$\mathcal{L}_2\phi_0 + \lambda_0^2\phi_0 = 0 \tag{39}$$

$$\mathcal{L}_2\phi_1 + \lambda_0^2\phi_1 = -2\lambda_0\lambda_1\phi_0 \tag{40}$$

subject to the boundary conditions (36) and (37).

Equation (39) subject to the condition (36) is an eigenvalue problem which yields an infinite set of eigenvalues λ_{0mn} and eigenfunctions ϕ_{0mn}. Equation (40) subject to the boundary condition (37) can then be solved by expressing ϕ_1 as a linear combination of these eigenfunctions. The solvability condition for this problem enables us to evaluate λ_1 and hence the value of λ to $O(\epsilon)$ as expressed by eqn (38).

4. SPECIAL CASE

In the case of uniformly prestressed circular plates with rectangular orthotropy, the plane stress solutions are:

$$N^*_{rr} = N^*_{\theta\theta} = N^*$$

$$N^*_{r\theta} = 0.$$

That is, $N_{rr} = N_{\theta\theta} = 1$ and $N_{r\theta} = 0$. In this case the $\mathcal{L}_2$ operator reduces to the Laplacian operator ∇^2 as seen in equation (5). Then, eqns (39) and (40) may be written as:

$$\nabla^2\phi_0 + \lambda_0^2\phi_0 = 0 \tag{41}$$

$$\nabla^2\phi_1 + \lambda_0^2\phi_1 = -2\lambda_0\lambda_1\phi_0. \tag{42}$$

The solution of eqn (41) subject to the boundary condition (36) can be written as

$$\phi_0 = bJ_m(\lambda_{0mn}r)\{A\exp(im\theta) + \bar{A}\exp(-im\theta)\} \tag{43}$$

where A is an arbitrary complex constant, λ_{0mn} is the nth root of

$$J_m(\lambda_0) = 0 \tag{44}$$

and b is chosen such that

$$b^2 \int_0^1 rJ_m^2(\lambda_{0mn}r)\,dr = 1$$

or

$$b^2 = 2[J_m'(\lambda_{0mn})]^{-2}. \tag{45}$$

Since the above eigenvalue problem has a nontrivial solution, the inhomogeneous eqn (42) subject to the inhomogeneous boundary condition (37) has a nontrivial solution if, and only if, a solvability condition is satisfied. This solvability condition furnishes the value of λ_1. To determine this, we express ϕ_1 as:

$$\phi_1(r, \theta) = \sum_{s=-\infty}^{\infty} \psi_s(r)\exp(is\theta). \tag{46}$$

Substituting for ϕ_0 and ϕ_1 into eqns (42) and (37), we have:

$$\sum_{s=-\infty}^{\infty}\left[\psi_s''+\frac{1}{r}\psi_s'+\left(\lambda_{0mn}^2-\frac{s^2}{r^2}\right)\psi_s\right]\exp(is\theta)=-2\lambda_{0mn}\lambda_1 b\,J_m(\lambda_{0mn}r)\{A\exp(im\theta)+\bar{A}\exp(-im\theta)\} \tag{47}$$

$$\sum_{s=-\infty}^{\infty}\psi_s(1)\exp(is\theta)=\frac{b\lambda_{0mn}}{\mu(\theta)}J_m'(\lambda_{0mn})\{A\exp(im\theta)+\bar{A}\exp(-im\theta). \tag{48}$$

Multiplying eqns (47) and (48) by $\exp(-im\theta)$ and integrating the results from $\theta=0$ to $\theta=2\pi$, we obtain

$$\psi_m''+\frac{1}{r}\psi_m'+\left(\lambda_{0mn}^2-\frac{m^2}{r^2}\right)\psi_m=-2\lambda_{0mn}\lambda_1 bAJ_m(\lambda_{0mn}r) \tag{49}$$

$$\psi_m(1)=b\lambda_{0mn}J_m'(\lambda_{0mn})\{Ag_0+\bar{A}g_{2m}\} \tag{50}$$

where

$$g_0=\frac{1}{2\pi}\int_0^{2\pi}\frac{1}{\mu(\theta)}\,d\theta$$

and

$$g_{2m}=\frac{1}{2\pi}\int_0^{2\pi}\frac{\exp(-i2m\theta)}{\mu(\theta)}\,d\theta. \tag{51}$$

The problem of determining the solvability condition for eqns (42) and (37) is thus reduced to that of determining the solvability condition of eqns (49) and (50). To accomplish this, multiply eqn (49) by $ru(r)$, integrate the result by parts from $r=0$ to $r=1$, and obtain

$$[ru(r)\psi_m'-ru'(r)\psi_m]_0^1+\int_0^1\psi_m(r)\left[(ru')'+\left(\lambda_{0mn}^2r-\frac{m^2}{r}\right)u\right]dr=-2\lambda_{0mn}\lambda_1Ab^2\int_0^1 rJ_m^2(\lambda_{0mn}r)\,dr.$$

We choose $u(r)$ to be a solution of the adjoint homogeneous problem; that is

$$u(r)=bJ_m(\lambda_{0mn}r). \tag{52}$$

Then,

$$[ru\psi_m'-ru'\psi_m]_0^1=-2\lambda_{0mn}\lambda_1A.$$

Using eqn (50) and the fact that $u(1)=0$, we obtain

$$-\{b\lambda_{0mn}J_m'(\lambda_{0mn})\}^2\{Ag_0+\bar{A}g_{2m}\}=-2\lambda_{0mn}\lambda_1A$$

or

$$\lambda_1=\lambda_{0mn}\{g_0+\bar{A}A^{-1}g_{2m}\}. \tag{53}$$

Hence

$$\lambda_{mn}^2=\lambda_{0mn}^2\{1+2\epsilon[g_0+\bar{A}A^{-1}g_{2m}]\}+O(\epsilon^2). \tag{54}$$

As $\lambda=\alpha\omega$ we have

$$\begin{aligned}\alpha\omega_{mn}=\lambda_{mn}&=\lambda_{0mn}[1+2\epsilon(g_0+\bar{A}A^{-1}g_{2m})]^{1/2}+\cdots\\&\approx\lambda_{0mn}[1+\epsilon(g_0+\bar{A}A^{-1}g_{2m})]+\cdots.\end{aligned} \tag{55}$$

Here λ_{0mn} is the nth zero of the Bessel function of the first kind and mth order.

Letting

$$A=\frac{1}{2}\gamma\exp(i\beta)\quad\text{and}\quad g_{2m}=\sigma_{2m}\exp(i\tau) \tag{56}$$

where γ, β, σ_{2m}, and τ are real constants, in eqn (55), we have

$$\alpha\omega_{mn} \approx \lambda_{0mn}\{1 + \epsilon[g_0 + \sigma_{2m} \exp(i[\tau - 2\beta])]\} + \cdots. \quad (57)$$

As ω_{mn} have to be real for a stable solution, we require

$$(\tau - 2\beta) = 0 \quad \text{or} \quad \pi. \quad (58)$$

That is, $\beta = 1/2\tau$, or $\beta = 1/2(\tau - \pi)$. This leads to the splitting of the frequency of vibration of the plate. Substituting eqn (58) into eqn (57), we have

$$\alpha\omega_{mn} \approx \lambda_{0mn}[1 + \epsilon(g_0 \pm \sigma_{2m})] + \cdots. \quad (59)$$

The solution (59) reduces to that in [5] for an isotropic plate.

The solutions for a boron-epoxy plate are obtained for high prestress values using the perturbation solution (59). These are compared with finite-element solutions. Table 1 shows good agreement between the two sets of results for $\epsilon \approx 0.1$. As ϵ decreases from this value the accuracy of the finite-element solution tends to deteriorate. On the other hand, as ϵ increases from 0.1, the accuracy of the numerical-perturbation technique tends to deteriorate.

Table 1. Correlation between perturbation solutions and finite-element solutions for a boron-epoxy circular plate

ϵ	$\alpha\omega_{mn}$ Finite-element method	$\alpha\omega_{mn}$ Numerical-perturbation method	% Difference
0.01	2.5062	2.4337	2.89
0.025	2.5313	2.4770	2.42
0.05	2.5841	2.5493	1.35
0.1	2.7055	2.6938	0.40
0.2	2.9987	2.9827	0.53
0.3	3.3526	3.2717	2.41
0.4	3.7563	3.5606	5.21
0.5	4.1991	3.8496	8.32
0.6	4.6729	4.1386	11.40
0.8	5.6887	4.7165	17.10

5. CONCLUSIONS

A numerical-perturbation technique has been proposed for the analysis of the transverse vibrations of highly prestressed anisotropic plates of circular geometry. Excellent correlation is observed between the numerical-perturbation solutions and the finite-element solutions for a boron-epoxy plate. This analysis of circular plates with rectangular orthotropy can accomodate any general anisotropic plate with known variations of the material properties. Furthermore, the solutions for annular anisotropic plates would form the basis for solutions to transverse vibrations of near circular, highly prestressed, annular plates [10].

REFERENCES

1. A. W. Leissa, *Vibration of Plates.* NASA SP-160 (1969).
2. J. E. Ashton and J. M. Whitney, *Theory of Laminated Plates.* Technomic Publishing Co. (1970).
3. O. C. Zienkiewicz, *The Finite Element Method in Engineering Science.* McGraw-Hill, New York (1971).
4. A. H. Nayfeh, *Perturbation Methods.* Interscience, New York (1973).
5. A. H. Nayfeh and M. P. Kamat, A numerical-perturbation technique for the transverse vibrations of highly prestressed plates. *J. Acoust. Soc. Am.* **61**(1), 95 (1977).
6. J. L. Sanders, Jr., Nonlinear theories for thin shells. *Quart. Appl. Math.* **21**, 21 (1963).
7. W. E. Alzheimer and R. T. Davis, Unsymmetrical bending of prestressed annular plates. *J. Engrg. Mech. Div. Proc. ASCE* **94**, 905 (1968).
8. W. B. Fraser, Bending of a highly stretched plate containing an eccentrically plate-reinforced circular hole. *Int. J. Solids Structures* **11**, 501 (1975).
9. W. B. Fraser, Bending of a stretched plate containing an eccentrically plate reinforced hole of arbitrary shape. *Int. J. Solids Structures* **11**, 1245 (1975).
10. A. H. Nayfeh, D. T. Mook, D. W. Lobitz and S. Sridhar, Vibrations of nearly annular and circular plates. *J. Sound & Vibration* **47**, 75 (1976).